Thermocapillary long waves in a liquid film flow. Part 1. Low-dimensional formulation

By C. RUYER-QUIL[1], B. SCHEID[1,2],
S. KALLIADASIS[3]†, M. G. VELARDE[4]
AND R. Kh. ZEYTOUNIAN[5]

[1]Laboratoire FAST, UMR 7608, CNRS, Universités P. et M. Curie et Paris Sud, Bât. 502,
Campus Universitaire, 91405 Orsay Cedex, France

[2]Service de Chimie-Physique E.P., Université Libre de Bruxelles, C.P. 165/62, 1050 Brussels, Belgium

[3]Department of Chemical Engineering, University of Leeds, Leeds LS2 9JT, UK

[4]Insituto Pluridisciplinar, Universidad Complutense de Madrid, Paseo Juan XXIII, n. 1,
E-28040 Madrid, Spain

[5]Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Asq cédex, France
ruyer@fast.u-psud.fr; bscheid@ulb.ac.be; s.kalliadasis@imperial.ac.uk;
velarde@fluidos.pluri.ucm.es; zeytounian@aol.com

We consider the dynamics of a thin liquid film falling down a uniformly heated wall. The heating sets up surface tension gradients that induce thermocapillary stresses on the free surface, thus affecting the evolution of the film. We model this thermocapillary flow by using a gradient expansion combined with a Galerkin projection with polynomial test functions for both velocity and temperature fields. We obtain equations for the evolution of the velocity and temperature amplitudes at first- and second-order in the expansion parameter. These equations are fully compatible close to criticality with the Benney long-wave expansion. Models of reduced dimensionality for the evolution of the local film thickness, flow rate and interfacial temperature only, are proposed.

1. Introduction

The formation of waves at the surface of a falling film heated from below can result from different mechanisms. The first one is the classical long-wave instability mode of an isothermal falling film first described extensively in the experimental work by Kapitza & Kapitza (1949) and which will be referred to hereafter as the *hydrodynamic mode*. The threshold of this mode and its linear stability properties were scrutinized by Benjamin (1957). Two additional instability modes have been found by Goussis & Kelly (1991). Both of them originate from the interfacial stress generated at the interface by the surface tension gradient. This is the well-known *Marangoni effect* (Colinet, Legros & Velarde 2001; Velarde & Zeytounian 2002) and the associated modes of instability will be referred to as the *thermocapillary modes*. The first thermocapillary instability mode was obtained by Pearson (1958) who analysed the stability of a horizontal layer with a non-deformable free surface, and is a short-wave mode. The second mode is a long-wave variety and was obtained by

† Present address: Department of Chemical Engineering, Imperial College, London SW7 2AZ, UK.

Scriven & Sternling (1964) who allowed the free surface of a horizontal layer to deform. However, Sternling & Scriven neglected gravity. The occurrence of this long-wave instability mode in the presence of gravity was then confirmed by Smith (1966).

Following the terminology adopted by Goussis & Kelly (1991), the hydrodynamic mode of instability will be referred to as the H-mode whereas the long-wave thermocapillary mode obtained by Scriven & Sternling and Smith will be denoted as the S-mode. The short-wave thermocapillary mode described by Pearson is not relevant in the present study which is devoted to the problem of long-wave instabilities on the surface of a film falling down a uniformly heated plane.

The large ratio between a typical wavelength of the instability and the average film thickness allows a gradient expansion of the velocity and temperature fields and the subsequently obtaining of systems of equations of reduced dimensionality. In the region where inertia is not important, namely if both Péclet and Reynolds numbers are of $O(1)$ or smaller, the temperature and velocity fields are slaved to the kinematics of the free surface and a single evolution equation for the film thickness h can be derived. This is effectively an extension of Benney's long-wave expansion (Benney 1966) (see also the review by Oron, Davis & Bankoff 1997) to non-isothermal flows and it was done by Joo, Davis & Bankoff (1991) who included, in addition to thermocapillary effects, evaporation and intermolecular forces. In the absence of these additional effects, their evolution equation will be referred to hereafter as the JDB equation.

For isothermal films, Benney's approach is exact in the limit of small Reynolds numbers but it breaks down at an $O(1)$ Reynolds number with turning points and branch multiplicity for the solitary-like wave solutions leading to a non-physical finite-time blow-up (Pumir, Manneville & Pomeau 1983; Scheid *et al.* 2005*b*). The same is true in the non-isothermal case. This was shown by Kalliadasis *et al.* (2003*a*) who traced the solitary wave solution branch of the JDB equation. Moreover, these authors adopted an integral-boundary-layer (IBL) approach for the equations of motion and energy equation. In the isothermal case, this formulation combines the assumption of a self-similar parabolic velocity profile beneath the film with the Kármán–Polhausen averaging method in boundary-layer theory. The approach was first suggested by Kapitza to describe stationary waves and later on extended by Shkadov and coworkers to non-stationary and three-dimensional films (Kapitza & Kapitza 1949; Shkadov 1967; Demekhin, Kaplan & Shkadov 1987; Demekhin & Shkadov 1984). The IBL model does not suffer from the shortcomings of Benney's expansion and performs well in the region of moderate Reynolds numbers and without any singularities for the solitary wave solution branch. For the non-isothermal case, Zeytounian (1998) derived an IBL model consisting of three equations in terms of the local film thickness (h), flow rate (q) and mean temperature across the layer. However, since the coupling between the temperature and velocity fields arises through the tangential stress balance at the interface, it is more appropriate to choose a weighted residuals approach for the energy equation that would put the emphasis on the interfacial temperature (θ) so that points near the interface have a 'larger weight' than points near the solid boundary. Such a formulation was proposed by Kalliadasis *et al.* (2003*a*) who adopted a linear test function for the temperature combined with a weighted residuals approach for the energy equation and obtained a three-equations model for h, q and θ.

However, despite the success of IBL in the nonlinear regime, it does not predict very accurately neutral and critical conditions and introduces an error, typically of the order of 20 %, for the critical Reynolds number. Hence the model suffers from the

same limitations with as Shkadov IBL model which also does not predict accurately the behaviour of the film close to criticality. In this case, the discrepancy is simply due to the velocity profile assumed in the Shkadov method: a self-similar parabolic profile across the film. Although this profile seems to be in agreement with the experiments by Alekseenko, Nakoryakov & Pokusaev (1994), and hence does capture most of the physics, corrections to the profile, known to exist at first order in the film parameter from the long-wave expansion, are important for an accurate prediction of the linear instability threshold.

Our purpose here is to overcome the limitations of the model equations derived by Kalliadasis *et al.* (2003*a*). In addition, we wish to introduce the second-order dissipative effects that are known to determine the amplitude of the front-running capillary waves in the case of isothermal flows (Ruyer-Quil & Manneville 2000, 2002). These second-order viscous terms were neglected in the formulation by Kalliadasis *et al.* (2003*a*) but they play an important role in the dispersion of waves for larger Reynolds numbers and/or smaller Kapitza numbers. The procedure followed here is effectively an extension of the methodology applied in the case of isothermal flows by Ruyer-Quil & Manneville (2000, 2002) and is based on a high-order weighted residuals approach with polynomial expansions for both velocity and temperature fields. Our investigation consists of two parts. Part 1 is dedicated to the formulation and derivation of a model which close to criticality is fully compatible with the long-wave classical expansion at first- and second-order in the film parameter. Hence, this equation corrects the deficiencies of the IBL approach for the heated falling film problem.

The paper is organized as follows. The basic equations, formulation and relevant dimensionless groups are given in §2. In §3 we truncate our basic equations to obtain the so-called 'boundary-layer' equations in which the inertia terms in the cross-stream momentum equation are neglected at first order in the film parameter. Section 4 outlines the weighted residuals approach for both velocity and temperature fields. This leads to two sets of model equations at first and second order in the film parameter. However, these equations are complicated and hence not convenient for practical applications. A procedure is then described in §5 that enables us to simplify our set of equations and to formulate models fully compatible with the Benney expansion up to second order. This approach results in systems of three coupled nonlinear partial differential equations for the evolution of the local film thickness, flow rate and interfacial temperature. The final system of evolution equations is obtained in §6 based on the idea that the second-order inertial terms should be written in a form as close as possible to the first-order ones in order to ensure convergence of the model for the widest possible range of parameters. Other formulations and models have been tested and compared; however, they do not perform as well as the proposed model (6.6) and they are not presented here for clarity and simplicity and in order to focus on the case that is ultimately analysed. Finally, a conclusion is offered in §7.

In Part 2 (Scheid *et al.* 2005*a*) we examine in detail the linear stability properties of the model derived in Part 1 and in particular we compare its neutral stability curves with those obtained from the Orr–Sommerfeld eigenvalue problem of the full Navier–Stokes/energy equations. We then compute nonlinear waves far from criticality. Particular emphasis is given to solitary waves. We scrutinize the influence of the thermocapillary effect on the instability and the properties of the solitary waves and we analyse the effect of Reynolds, Prandtl and Marangoni numbers on the shape of the waves, flow patterns and temperature distributions in the film.
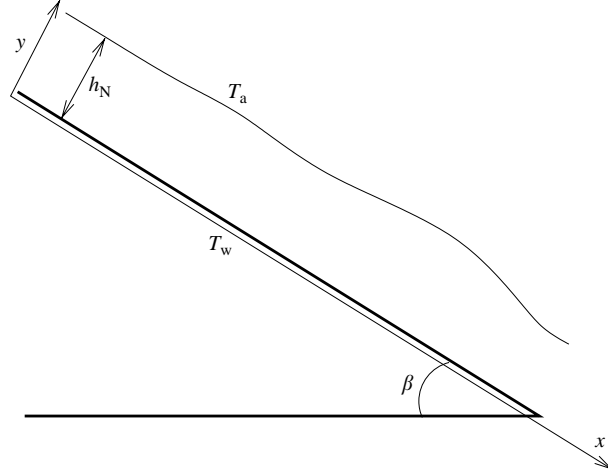
FIGURE 1. Sketch of the profile geometry for a film falling down a uniformly heated plane. h_N is the Nusselt flat film thickness.

2. Problem formulation

We consider a film falling down a uniformly heated inclined plane with inclination angle β with respect to the horizontal direction. Figure 1 sketches the flow situation. We introduce a Cartesian coordinate system with x the streamwise coordinate in the flow direction and y the coordinate normal to the substrate; g denotes the gravitational acceleration, μ and ρ the dynamic viscosity and density of the liquid, respectively. The ambient gas phase is air at temperature T_a. The wall is maintained at temperature T_w ($>T_a$).

The variation of surface tension with temperature is modelled by the linear approximation, $\sigma(T) = \sigma(T_0) - (-\mathrm{d}\sigma/\mathrm{d}T)|_{T_0}(T - T_0)$, where the rate $\mathrm{d}\sigma/\mathrm{d}T|_{T_0}$ is < 0 for typical fluids. The reference temperature T_0 can be either the interfacial temperature of the basic state, i.e. the flat film, or the air temperature. The length and time scales are obtained from the streamwise gravitational acceleration $g \sin \beta$ and the kinematic viscosity $\nu = \mu/\rho$ which yields $l_0 = \nu^{2/3}(g \sin \beta)^{-1/3}$ and $t_0 = \nu^{1/3}(g \sin \beta)^{-2/3}$ so that the velocity and pressure scales are $U_0 = l_0 t_0^{-1} = (\nu g \sin \beta)^{1/3}$ and $P_0 = \rho(\nu g \sin \beta)^{2/3}$. These scales express the importance of the viscous and gravitational forces in our system. Here we consider inclined planes for which $\sin \beta$ is of order unity and film flows of thicknesses h_N of the order of the length scale l_0. Finally, for the temperature field we introduce the non-dimensionalization, $\bar{T} = (T - T_a)/(T_w - T_a)$, so that the dimensionless wall and air temperatures are $\bar{T} = 1$ and $\bar{T} = 0$, respectively, where the bars denote dimensionless variables.

The dimensionless momentum equation then is

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \boldsymbol{i} - \cot \beta \, \boldsymbol{j} + \nabla^2 \boldsymbol{u}, \tag{2.1}$$

where $\boldsymbol{u}$ and p are the dimensionless velocity and pressure fields, respectively. We also have the continuity equation

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.2}$$

the no-slip boundary condition at the plane

$$\boldsymbol{u}|_{y=0} = 0, \tag{2.3}$$

the kinematic condition at the free surface

$$(\partial_t + \boldsymbol{u} \cdot \nabla)(h - y) = 0, \tag{2.4}$$

where h is the local film thickness, and continuity of stress at the free surface

$$-p\boldsymbol{n} + 2\boldsymbol{d} \cdot \boldsymbol{n} = -(\Gamma - MaT)\nabla \cdot \boldsymbol{n} - Ma(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}) \cdot \nabla T \cdot (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}), \tag{2.5}$$

where the bars used to distinguish between dimensional and dimensionless temperatures have been dropped. Here $\boldsymbol{n}$ is the outward-pointing normal at the interface, $\boldsymbol{d} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$ and $\boldsymbol{I}$ the identity tensor. Surface tension effects are measured by the Kapitza number

$$\Gamma \equiv \frac{\sigma(T_a)}{\rho l_0{}^2 g \sin \beta} = \frac{\sigma(T_a)}{\rho \nu^{4/3}(g \sin \beta)^{1/3}},$$

and the Marangoni number

$$Ma \equiv \Gamma \frac{-\mathrm{d}\sigma/\mathrm{d}T|_{T_0}(T_w - T_a)}{\sigma(T_a)} = \frac{-\mathrm{d}\sigma/\mathrm{d}T|_{T_0}(T_w - T_a)}{\rho \nu^{4/3}(g \sin \beta)^{1/3}}.$$

We also have the energy equation

$$Pr(\partial_t T + \boldsymbol{u} \cdot \nabla T) = \nabla^2 T, \tag{2.6}$$

which is coupled to the hydrodynamic equations through the stress balance in (2.5). Here $Pr = \nu/\kappa$ denotes the Prandtl number with κ the thermal diffusivity. Our system is finally completed with the wall boundary condition

$$T|_0 = 1, \tag{2.7}$$

and Newton's law of cooling at the interface

$$-\nabla T|_h \cdot \boldsymbol{n} = Bi\, T|_h, \tag{2.8}$$

where the Biot number is defined as $Bi = q_0 l_0/K$ with $K = \rho c_p \kappa$ the thermal conductivity of the liquid and c_p the constant-pressure heat capacity and with q_0 a heat transfer coefficient that describes the rate of heat transport from the liquid to the ambient gas phase.

Finally, we note that the full system of Navier–Stokes/energy equations and wall/free-surface boundary conditions in (2.1)–(2.8) has a trivial solution that corresponds to a flat film of dimensionless thickness h_N with a parabolic velocity distribution and a linear temperature distribution, namely,

$$u = yh_N - \tfrac{1}{2}y^2, \quad T = 1 - \frac{Bi}{1 + Bih_N}y. \tag{2.9}$$

3. Boundary-layer equation

We now assume that the typical length of the waves is large in comparison with the thickness h_N of the film so that the slope of the interface $\partial_x h$ is always small. Defining a formal film parameter $\epsilon \sim \partial_x$, ∂_t, we shall investigate slow time and space modulations of the basic flat-film solution (2.9).

In order to capture dispersion effects induced by streamwise viscous terms that were seen to play an important role in the isothermal case (Ruyer-Quil & Manneville 2002), we keep terms up to $O(\epsilon^2)$ in the derivation process. As in boundary-layer theory, we neglect the inertial terms in the cross-stream momentum equation and

by integrating along y we obtain, with the normal stress condition at the interface, an approximation at $O(\epsilon)$ of the pressure distribution in the liquid and therefore an approximation at $O(\epsilon^2)$ of the pressure gradient in the streamwise direction

$$\partial_x p = -\partial_{xx}u - \partial_x[\partial_x u|_h] + \cot\beta\,\partial_x h - \Gamma\partial_{xxx}h, \qquad (3.1)$$

where the continuity condition

$$\partial_x u + \partial_y v = 0 \qquad (3.2)$$

has been used. It is useful to note that all neglected terms in (3.1) are at least of $O(\epsilon^3)$ but we have kept the main contribution of surface tension which is also formally of $O(\epsilon^3)$. As awkward as it may appear, this 'inconsistency' is necessary. Indeed, in the isothermal case it is this term related to capillary forces that prevents the waves from breaking. By now substituting equation (3.1) into the streamwise momentum equation, one obtains

$$\partial_t u + u\partial_x u + v\partial_y u - (\partial_{yy} + 2\partial_{xx})u - 1 + \cot\beta\,\partial_x h - \partial_x[\partial_x u|_h] - \Gamma\partial_{xxx}h = 0, \quad (3.3)$$

which will be referred to as the *second-order boundary-layer equation*. In addition, we have the two-dimensional energy equation

$$Pr(\partial_t T + u\partial_x T + v\partial_y T) = (\partial_{xx} + \partial_{yy})T. \qquad (3.4)$$

Equations (3.2)–(3.4) are completed by the two-dimensional kinematic condition at the interface (which is exact for all orders)

$$\partial_t h + u\partial_x h = v|_h, \qquad (3.5)$$

the continuity of the tangential stress truncated at $O(\epsilon^2)$

$$\begin{aligned}
\partial_y u|_h &= 4\partial_x h\partial_x u|_h - \partial_x v|_h - Ma[(\partial_x + \partial_x h\partial_y)T]|_h \\
&= 4\partial_x h\partial_x u|_h - \partial_x v|_h - Ma\partial_x[T|_h],
\end{aligned} \qquad (3.6)$$

the truncated heat balance at the interface

$$\partial_y T|_h = -Bi\big(1 + \tfrac{1}{2}(\partial_x h)^2\big)T|_h + \partial_x h\partial_x T|_h, \qquad (3.7)$$

and the Dirichlet conditions at the wall (2.3) and (2.7).

4. Weighted residuals approach

We now apply the polynomial expansion approach developed by Ruyer-Quil & Manneville (2000) for isothermal films. The basic idea is to separate the variables and to expand the velocity and temperature fields on a set of test functions depending on the reduced coordinate $\hat{y} = y/h$ which is a natural similarity variable as it converts the boundary-value problem in the interval $[0, h]$ to a problem in $[0, 1]$. To satisfy the boundary conditions (2.3) and (2.7) defining the velocity and temperature distributions at the wall, we write

$$u(x, y, t) = \sum_{i=0}^{i_{max}} a_i(x, t)f_i(y/h(x, t)), \quad T(x, y, t) = 1 + \sum_{i=0}^{i_{max}} b_i(x, t)g_i(y/h(x, t)), \quad (4.1)$$

where $f_i(0) = g_i(0) = 0$.

It is appropriate to choose polynomial test functions f_i, g_i for at least two reasons: (i) our analysis is based on the assumption of slow modulations of the basic-state solution which corresponds to a parabolic velocity profile and a linear temperature

distribution, so that it is necessary to introduce the flat-film solution (2.9) into the expansion; (ii) polynomials form a closed set with respect to differentiations and products appearing in (3.2)–(3.7). We therefore choose $f_0 = \hat{y} - \frac{1}{2}\hat{y}^2$ and $g_0 = \hat{y}$ corresponding to the basic state (2.9) and complete the set of test functions with

$$f_1(\hat{y}) = \hat{y}, \qquad f_i(\hat{y}) = \hat{y}^{i+1}, \; i \geqslant 2 \quad \text{and} \quad g_i(\hat{y}) = \hat{y}^{i+1}, \; i \geqslant 1, \tag{4.2}$$

to obtain the polynomial bases for the projection. Note that the Dirichlet conditions (2.3) and (2.7) are then automatically satisfied.

Since $2i_{\max} + 3$ unknowns have been introduced, namely h, a_i and b_i, $2i_{\max} + 3$ equations should be written to determine them. The first one is the kinematic condition at the interface (3.5) which can be replaced by integrating the continuity equation (3.2) along the normal coordinate to give

$$\partial_t h + \partial_x q = 0, \tag{4.3}$$

where $q = \int_0^h u \, dy$ is the flow rate in the streamwise direction. Two additional equations are the boundary conditions (3.6)–(3.7). Defining $2i_{\max}$ weight functions $w_j(\hat{y})$, the final equations and closure are obtained by the vanishing residuals

$$\mathcal{R}_q(w_j) \equiv \int_0^h w_j(\hat{y})[\partial_t u + u\partial_x u + v\partial_y u - (\partial_{yy} + 2\partial_{xx})u$$
$$- 1 + \cot\beta \, \partial_x h - \partial_x[\partial_x u|_h] - \Gamma \partial_{xxx} h] \, dy = 0, \tag{4.4a}$$

$$\mathcal{R}_T(w_{i_{\max}+j}) \equiv \int_0^h w_{i_{\max}+j}(\hat{y})[Pr(\partial_t T + u\partial_x T + v\partial_y T) - (\partial_{xx} + \partial_{yy})T] \, dy = 0, \tag{4.4b}$$

with $0 \leqslant j \leqslant i_{\max} - 1$ and where u and T are given by the expansions (4.1) and v from the continuity equation, $v = -\int_0^y \partial_x u \, dy$.

At this point, the method we are using is simply one of the numerous weighting residual strategies which differ from each other only by the specific choice of the weights w_j. As pointed out in previous studies dealing with the isothermal case (see Ruyer-Quil & Manneville 2000, 2002), it is not necessary to specify the weighting residual method we are applying. Indeed, requiring equations (3.3) and (3.4) to be satisfied everywhere – and not simply on average – and inserting into these equations the expansions (4.1), (4.2), leads to the cancellation of two polynomials in the reduced normal coordinate $\hat{y}$. Then, it can be proved by examining the order of magnitude with respect to ϵ of each term in (3.3) and (3.4), that the number of independent conditions on the unknowns a_i and b_i provided by the cancellation of these two polynomials is equal to the number of the residuals (4.4) if $i_{\max}$ is chosen large enough (Ruyer-Quil & Manneville 2002). In this case, any choice of the weight functions would lead to equivalent systems of equations and then to the same reduced model for the dynamics of the flow. Nevertheless, it is important to emphasize that we are not simply applying a numerical method. Our approach is rather to combine a classic averaging method with a perturbation technique to the flat-film basic state (2.9) corresponding to $a_0 = h^2$, $b_0 = -Bih/(1 + Bih)$, $a_i = b_i = 0$, $i \geqslant 1$. Terms of order higher than ϵ^2 will be neglected during the derivation process.

4.1. *Formulation at first order*

To illustrate our procedure, we give the formulation consistent at $O(\epsilon)$ with all terms of higher order neglected. This is a helpful step for the choice of the simplest methodology to use for the projection of the velocity and temperature fields onto the amplitudes of the polynomials appearing at first order. Thus (3.3), (3.4) are simplified

to

$$\partial_t u + u\partial_x u + v\partial_y u = \partial_{yy}u + 1 - \cot\beta\,\partial_x h + \Gamma\partial_{xxx}h, \qquad (4.5a)$$
$$Pr(\partial_t T + u\partial_x T + v\partial_y T) = \partial_{yy}T. \qquad (4.5b)$$

This set of equations is completed by the continuity condition (3.2) and the boundary conditions (2.3), (2.7), the integral version of the kinematic condition (4.3), the continuity of the tangential stress at the surface (3.6) which truncated at $O(\epsilon)$ simply reads

$$\partial_y u|_h = -Ma\partial_x[T|_h], \qquad (4.6)$$

and finally the heat balance at the interface

$$\partial_y T|_h = -BiT|_h. \qquad (4.7)$$

The residuals are now simplified to

$$\int_0^h w_j(\hat{y})[\partial_t u + u\partial_x u + v\partial_y u - \partial_{yy}u]\mathrm{d}y + h[-1 + \cot\beta\,\partial_x h - \Gamma\partial_{xxx}h]\int_0^1 w_j(\hat{y})\,\mathrm{d}\hat{y} = 0, \qquad (4.8a)$$

$$\int_0^h w_{i_{\max}+j}(\hat{y})[Pr(\partial_t T + u\partial_x T + v\partial_y T) - \partial_{yy}T]\mathrm{d}y = 0, \qquad (4.8b)$$

where the contributions from surface tension effects have been kept. The amplitudes a_i and b_i, $i \geqslant 1$, result from the slow space and time modulations of the free surface so that they are at least first-order quantities in ϵ. Therefore, the space and time derivatives of a_i and b_i, $i \geqslant 1$, are negligible. One then is led to a linear system for a_i and b_i whose coefficients depend at most on a_0, b_0, h and with a right-hand side that depends on h, a_0, b_0 and their derivatives:

$$\sum_{j'=1}^{2i_{\max}} \alpha_{jj'}A_{j'} = \beta_j(h, a_0, b_0, \partial_{x,t}h, \partial_{x,t}a_0, \partial_{x,t}b_0), \qquad 1 \leqslant j \leqslant 2i_{\max}, \qquad (4.9)$$

where $A_j \equiv a_j$ and $A_{i_{\max}+j} \equiv b_j$, $1 \leqslant j \leqslant i_{\max}$. Solving for the A_j leads to explicit formulations of the amplitudes a_j, b_j as functions of a_0, b_0, h and their derivatives. Thus, we obtain a set of three evolution equations for h, a_0 and b_0 modelling the entire dynamics of the film flow.

As was pointed out above, inserting the expansions (4.1), (4.2) in (4.5) leads to the cancellation of two polynomials in the reduced normal coordinate $\hat{y}$, say $\mathscr{P}(\hat{y})$ and $\mathscr{Q}(\hat{y})$, corresponding to the momentum and heat equation, respectively. Because the advection terms $\partial_t u + u\partial_x u + v\partial_y u$ and $\partial_t T + u\partial_x T + v\partial_y T$ are first-order quantities, their truncation at $O(\epsilon)$ involves only the parabolic and linear profiles corresponding to a_0 and b_0. Consequently, the advection terms are polynomials in $\hat{y}$ of degree four and three only. Therefore, the monomials of highest degree appearing in $\mathscr{P}(\hat{y})$ and $\mathscr{Q}(\hat{y})$ originate from the terms $\partial_{yy}u$ and $\partial_{yy}T$ so that $\mathscr{P}(\hat{y})$ and $\mathscr{Q}(\hat{y})$ are of degree $i_{\max} - 1$. Cancelling those two polynomials gives $2i_{\max}$ independent relationships, i.e. the same as the number of residuals (4.8), so that they are equivalent systems of equations leading to the same evolution equations for h, a_0 and b_0. Because each different weighting residual technique only differs by its specific definitions for the weighting functions w_j, it is relevant to look for the best choice of w_j that would simplify the algebraic manipulations.

Let us first consider more specifically the residuals (4.8a). Because $\partial_t u + u\partial_x u + v\partial_y u$ are first-order terms, the unknowns a_i, b_i, $i \geqslant 1$, may enter into their evaluation only

through the integral $\int_0^h w_j \partial_{yy} u$. Two integrations by parts give

$$\int_0^h w_j \left(\frac{y}{h}\right) \partial_{yy} u \, dy = \left[w_j \left(\frac{y}{h}\right) \partial_y u \right]_0^h - \frac{1}{h} \left[w_j' \left(\frac{y}{h}\right) u \right]_0^h + \frac{1}{h^2} \int_0^h w_j'' \left(\frac{y}{h}\right) u \, dy. \tag{4.10}$$

As $\partial_y u|_h$ given by equation (4.6) is proportional to $\partial_x [T|_h]$, at first order it may only involve h, a_0 and b_0. Making also use of the no-slip condition on the plate, $u|_0 = 0$, only three terms are left to consider, namely $w_j(0)\partial_y u|_0$, $w_j'(1)u|_h$ and $\int_0^h w_j''(y/h)u \, dy$. This suggests choosing w_0 such that $w_0(0) = 0$, $w_0'(1) = 0$ and w_0'' a constant and to introduce the flow rate $q \equiv \int_0^h u \, dy$. This corresponds to the zeroth-order formulation of the problem for the velocity:

$$\partial_{yy} u = -1, \qquad u|_0 = 0, \qquad \partial_y u|_h = 0. \tag{4.11}$$

Considering also the two integrations by parts performed in (4.10), such a similitude is obviously related to the fact that the linear operator ∂_{yy} is self-adjoint in the space of functions satisfying the boundary conditions (4.11).

Now, it seems appropriate to link the amplitude of the parabolic profile a_0 to the flow rate, which is a physical quantity appearing explicitly in the integral form of the kinematic condition (4.3). To introduce q explicitly into our expansion let us integrate $u(x, y, t)$ in (4.1) between 0 and h to obtain the expression

$$a_0 = 3\frac{q}{h} - \frac{3}{2}a_1 - \sum_{i=2}^{i_{max}} \frac{3}{i+2}a_i. \tag{4.12}$$

Therefore, evaluating the residual (4.8a) corresponding to $j = 0$ with $w_0 \equiv f_0$ leads to

$$\frac{2}{5}\partial_t q - \frac{23}{40}\frac{q}{h}\partial_t h - \frac{18}{35}\frac{q^2}{h^2}\partial_x h + \frac{111}{280}\frac{q}{h}\partial_x q + \frac{q}{h^2}$$
$$+ \frac{1}{2}Ma\partial_x[T|_h] + \frac{1}{3}h[-1 + \cot\beta \, \partial_x h - \Gamma\partial_{xxx}h] = 0, \tag{4.13}$$

where the unknowns a_i do not appear. Choosing the weight functions to be the test functions themselves is the essence of the Galerkin method, which is equivalent to a variational method – whenever a variational formulation is available (Finlayson 1972).

Turning to the weighted residuals for the heat equation (4.8b) and with the same arguments, the unknowns a_i, b_i, $i \geqslant 1$, may only play a role through the integral $\int_0^h w_j \partial_{yy} T$

$$\int_0^h w_j \left(\frac{y}{h}\right) \partial_{yy} T \, dy = \left[w_j \left(\frac{y}{h}\right) \partial_y T \right]_0^h - \frac{1}{h} \left[w_j' \left(\frac{y}{h}\right) T \right]_0^h + \frac{1}{h^2} \int_0^h w_j'' \left(\frac{y}{h}\right) T \, dy. \tag{4.14}$$

Making use of the boundary condition at the surface (4.7) and the constant temperature distribution at the wall $T|_0 = 1$ we obtain

$$\int_0^h w_j \left(\frac{y}{h}\right) \partial_{yy} T \, dy = -Bi w_j(1)T|_h - w_j(0)\partial_y T|_0$$
$$+ \frac{1}{h}[w_j'(0) - w_j'(1)T|_h] + \frac{1}{h^2} \int_0^h w_j'' \left(\frac{y}{h}\right) T \, dy. \tag{4.15}$$

Following exactly the same approach as before would lead to the choice for the first weight function $w_{i_{max}}(0) = 0$, $w'_{i_{max}}(1)=0$ and setting $w''_{i_{max}}$ to a constant would introduce the average temperature across the flow, $(1/h)\int_0^h T\,\mathrm{d}y$. This choice would obviously be problematic since the term $w_{i_{max}}(1)T|_h$ would remain in (4.15) and in (4.13) through the Marangoni effect. On the other hand, it is the exchanged heat flux at the surface $\partial_y T|_h$, or the temperature at the surface $T|_h$, which have physical significance. Thus, because of its role in (4.13), we prefer to put the emphasis on $\theta \equiv T|_h$ by choosing $w_{i_{max}}(0) = 0$, $w''_{i_{max}} = 0$ so that $w_{i_{max}} \propto \hat{y} = g_0$. This choice has the obvious advantage of dissociating the coupling term $\frac{1}{2}Ma\partial_x[T|_h]$ in (4.13) from the definition of any other amplitudes needed to describe the temperature distribution. It is therefore appropriate to replace the physically meaningless unknown b_0 by θ through the substitution

$$b_0 = \theta - 1 - \sum_{i=1}^{i_{max}} b_i. \tag{4.16}$$

From the residual (4.8b) corresponding to $w_{i_{max}} \equiv g_0 = \hat{y}$, we then obtain

$$Pr\left[\frac{1-\theta}{3}\partial_t h + \frac{1}{3}h\partial_t\theta + \frac{11}{40}(1-\theta)\partial_x q + \frac{9}{20}q\partial_x\theta\right] + \frac{\theta-1}{h} + Bi\,\theta = 0. \tag{4.17}$$

Again, we find that the Galerkin method is the most effective one, requiring less algebra.

Using now the equivalence $\partial_t h = -\partial_x q$ given by the integral formulation of the kinematic condition at the surface (4.3), a model consistent at $O(\epsilon)$ can be formulated in terms of three coupled evolution equations for h, q and θ

$$\partial_t h = -\partial_x q, \tag{4.18a}$$

$$\partial_t q = \frac{5}{6}h - \frac{5}{2}\frac{q}{h^2} - \frac{17}{7}\frac{q}{h}\partial_x q + \left(\frac{9}{7}\frac{q^2}{h^2} - \frac{5}{6}\cot\beta\,h\right)\partial_x h - \frac{5}{4}Ma\partial_x\theta + \frac{5}{6}\Gamma h\partial_{xxx}h, \tag{4.18b}$$

$$Pr\partial_t\theta = 3\frac{[1-(1+Bih)\theta]}{h^2} + Pr\left[\frac{7}{40}\frac{(1-\theta)}{h}\partial_x q - \frac{27}{20}\frac{q}{h}\partial_x\theta\right]. \tag{4.18c}$$

The set of equations in (4.18) can be contrasted to the model derived by Kalliadasis *et al.* (2003a) and Kalliadasis, Kiyashko & Demekhin (2003b). The functional form of the first-order averaged heat equation (4.18c) is very similar to theirs (see the two-dimensional formulation of equation (4) in Kalliadasis *et al.* 2003a), since for the derivation of both equations, the temperature across the film is essentially a self-similar linear profile with the weight function for the energy equations chosen within the Galerkin framework, e.g. the first-order polynomial y/h. The two averaged heat equations (4.18c) and equation (4) in Kalliadasis *et al.* (2003a) differ only in the choice of the scaling for the temperature and the presence of the Prandtl number in (4.18c) instead of the Péclet number in equation (4) in Kalliadasis *et al.* (2003a). In fact, the two models really differ in the treatment of the momentum equation. Equation (4.18b) contains the same terms with the corresponding averaged momentum equation in Kalliadasis *et al.* (2003a, b) but with different coefficients. Apart from some numerical factors due to different scalings (e.g. a factor three appearing in equation (2a) in Kalliadasis *et al.* (2003a) due to a velocity scaling based on the averaged velocity of the flat film), these modifications originate from a more detailed description of the perturbed velocity field which is not limited in the present approach to remain parabolic as in Kalliadasis *et al.* (2003a, b) – this parabolic profile

is effectively the essence of the Shkadov IBL approach. The present approach for the momentum equation is based on a Galerkin projection for the velocity field with weight functions as the test functions themselves, unlike the Shkadov IBL treatment of the momentum equation in Kalliadasis *et al.* (2003*a*, *b*) which is effectively a weighted residuals approach using a single test function (the self-similar parabolic profile) and a weight function equal to 1.

4.2. *Formulation at second order*

The aim now is to make our model consistent at second order also, i.e. accounting for the ϵ^2-order viscous and thermal diffusion terms of the boundary layer/energy equations (3.3), (3.4). For this purpose, we require the solution of equation (4.9) for the amplitudes of the projections,

$$a_1 = -\frac{6}{5}h\partial_x\left[\frac{q^2}{h}\right] - h\partial_t q - Mah\partial_x\theta, \quad a_2 = q\partial_x q + \frac{1}{2}h\partial_t q, \quad (4.19a, b)$$

$$a_3 = -\frac{3}{4}\frac{q^2}{h}\partial_x h - \frac{1}{8}h\partial_t q, \quad a_4 = -\frac{3}{40}h^6\partial_x\left[\frac{q^2}{h^6}\right], \quad (4.19c, d)$$

$$a_5 = \frac{1}{80}h^6\partial_x\left[\frac{q^2}{h^6}\right], \quad (4.19e)$$

$$b_2 = \frac{1}{6}Prh\left((\theta-1)\partial_x q + h\partial_t\theta\right), \quad b_3 = \frac{1}{8}Prh\left(-(\theta-1)\partial_x q + 2q\partial_x\theta\right), \quad (4.19f, g)$$

$$b_4 = \frac{1}{40}Prh\left((\theta-1)\partial_x q - 3q\partial_x\theta\right), \quad (4.19h)$$

$$b_1 = 0, \quad a_i = b_j = 0, \quad i \geqslant 6, \; j \geqslant 5.$$

Note that the amplitudes a_i of the monomials of degree greater than or equal to seven are identically equal to zero at first order. This can be shown by examining the degree of the polynomial in $\hat{y}$ corresponding to the left-hand side of (4.5*a*). Because f_0 is of degree two, this polynomial is of degree four, so that the right-hand side of (4.5*a*) is also a polynomial of degree four. Hence, the amplitude a_n corresponding to $f_n = \hat{y}^{n+1}$ is equal to zero if $n \geqslant 6$, with the operator ∂_{yy} decreasing its degree by two. The same argument can be applied to (4.5*b*) where the inertial terms $\partial_t T + u\partial_x T + v\partial_y T$ at the left-hand-side are a polynomial in $\hat{y}$ of degree three only.

Consequently, the derivatives of the fields a_i, $i \geqslant 6$, b_j, $j \geqslant 5$, are of order higher than ϵ^2 and can be dropped at this stage of the approximation. Their dynamics is thus slaved to the dynamics of the other unknowns. From the expressions (4.19), we can now obtain $a_4 = -6a_5$, $a_2 = -4a_3 + 40a_5$ and $a_1 = 8a_3 - 96a_5 - Mah\partial_x\theta$ so that, eliminating these amplitudes in (4.12) yields $a_0 = (3q)/h - \frac{48}{5}a_3 + \frac{816}{7}a_5 + \frac{3}{2}Mah\partial_x\theta$. The velocity field at first order can then be written as

$$u = 3\frac{q}{h}f_0(\hat{y}) + Mah\partial_x\theta\tilde{f}_1(\hat{y}) + a_3\tilde{f}_3(\hat{y}) + a_5\tilde{f}_5(\hat{y}), \quad (4.20)$$

where $\tilde{f}_1 = -\frac{3}{4}\hat{y}^2 + \frac{1}{2}\hat{y}$, $\tilde{f}_3 = \hat{y}^4 - 4\hat{y}^3 + \frac{24}{5}\hat{y}^2 - \frac{8}{5}\hat{y}$ and $\tilde{f}_5 = \hat{y}^6 - 6\hat{y}^5 + 40\hat{y}^3 - \frac{408}{7}\hat{y}^2 + \frac{144}{7}\hat{y}$. Therefore, u is a combination of four independent fields q/h, a_3, a_5 and $h\partial_x\theta$ rather than six as would be expected at first. Similarly, T can be written at first order as a combination of four independent fields, namely θ, b_2, b_3 and b_4. As a consequence, a consistent formulation of a model for the dynamics of the flow at second order would require nine unknowns corresponding to the introduction of eight independent fields to correctly represent the temperature and velocity distributions, and of course the film thickness h.

Since the degree of the polynomials $\tilde{f}_1$, $\tilde{f}_3$, $\tilde{f}_5$ is smaller than or equal to six, the second-order dissipative term $\partial_{yy}u$ and the quadratic nonlinearities of the Navier–Stokes equation imply that the description of the velocity field at $O(\epsilon^2)$ involves polynomials of degree up to ten. Therefore, the set of test functions for the velocity field needs to be completed by six other functions in order to obtain a basis for the set of polynomials of degree up to ten satisfying the Dirichlet condition at the wall. Turning now to the modelling of the heat equation at second order, a basis for the set of polynomials of degree up to nine satisfying the Dirichlet condition is required to fully describe the temperature field at that order. This means that six corresponding amplitudes for the velocity field and five for the temperature field need to be eliminated (through a slaving principle) to obtain a set of eight evolution equations for the eight unknowns required to correctly describe the dynamics of the flow at second order (plus the conservation equation (4.3)). Needless to say, such a task would require a cumbersome calculation and hence a short cut would be welcome.

Following the same approach as in the isothermal case, let us construct a new set of polynomial test functions F_i satisfying the orthogonality condition $\int_0^1 F_i\,F_j\mathrm{d}\hat{y} \propto \delta_{ij}$ with the help of a Gram–Schmidt orthogonalization procedure so that $F_0 \equiv f_0$, F_1, F_2 and F_3 are linear combinations of f_0, $\tilde{f}_1$, $\tilde{f}_3$ and $\tilde{f}_5$. The result is

$$F_0 = \hat{y} - \frac{1}{2}\hat{y}^2, \tag{4.21a}$$

$$F_1 = \hat{y} - \frac{17}{6}\hat{y}^2 + \frac{7}{3}\hat{y}^3 - \frac{7}{12}\hat{y}^4, \tag{4.21b}$$

$$F_2 = \hat{y} - \frac{13}{2}\hat{y}^2 + \frac{57}{4}\hat{y}^3 - \frac{111}{8}\hat{y}^4 + \frac{99}{16}\hat{y}^5 - \frac{33}{32}\hat{y}^6, \tag{4.21c}$$

$$F_3 = \hat{y} - \frac{531}{62}\hat{y}^2 + \frac{2871}{124}\hat{y}^3 - \frac{6369}{248}\hat{y}^4 + \frac{29601}{2480}\hat{y}^5 - \frac{9867}{4960}\hat{y}^6. \tag{4.21d}$$

The functions F_1 and F_2 have been chosen so that they correspond exactly to the polynomials introduced in the isothermal case. The introduction of the polynomial F_3 is made necessary by the presence of the Marangoni effect which modifies the stress condition at the interface (4.6).

Similarly, a set of orthogonal test functions for the temperature field is constructed from linear combinations of g_0, g_2, g_3 and g_4 such that $G_0 \equiv g_0$:

$$G_0 = \hat{y}, \tag{4.22a}$$

$$G_1 = \hat{y} - \frac{5}{3}\hat{y}^3, \tag{4.22b}$$

$$G_2 = \hat{y} - 7\hat{y}^3 + \frac{32}{5}\hat{y}^4, \tag{4.22c}$$

$$G_3 = \hat{y} - \frac{56}{3}\hat{y}^3 + \frac{192}{5}\hat{y}^4 - 21\hat{y}^5. \tag{4.22d}$$

Therefore, the velocity field and the temperature field at $O(\epsilon)$ are given by

$$u = \frac{3}{h}(q - s_1 - s_2 - s_3)\,F_0(\hat{y}) + 45\frac{s_1}{h}F_1(\hat{y}) + 210\frac{s_2}{h}F_2(\hat{y}) + 434\frac{s_3}{h}F_3(\hat{y}), \tag{4.23a}$$

$$T = 1 + (\theta - 1 - t_1 - t_2 - t_3)G_0(\hat{y}) - \frac{3}{2}t_1 G_1(\hat{y}) + \frac{5}{2}t_2 G_2(\hat{y}) - \frac{15}{4}t_3 G_3(\hat{y}). \tag{4.23b}$$

In line with our previous derivation of a second-order consistent model for the isothermal case (Ruyer-Quil & Manneville 2000), the first-order fields s_i, $1 \leqslant i \leqslant 3$, have been introduced so that u preserves the definition of the flow rate q, $q = \int_0^h u\,\mathrm{d}y$,

as it should. These fields correspond to corrections to the amplitude of the parabolic velocity profile and at the same time their role in the velocity profile is similar to that of q so that the final evolution equations for q and s_i will have similar functional forms. In the same spirit, the introduction of the fields t_i, $1 \leqslant i \leqslant 3$, preserves the definition of the temperature at the surface $\theta = T|_{y=h}$. Note that G_0 and $-\frac{3}{2}G_1$ are Legendre polynomials. This is quite fortuitous as the Legendre polynomials form an orthogonal basis with respect to the scalar product $\int_{-1}^{1} \cdot \mathrm{d}\hat{y}$ instead of $\int_{0}^{1} \cdot \mathrm{d}\hat{y}$. To complete our set of test functions in order to obtain a basis for the set of polynomials of degree up to ten satisfying the no-slip condition, we now write

$$
\begin{aligned}
u = {} & \frac{3}{h}(q - s_1 - s_2 - s_3)F_0(\hat{y}) + 45\frac{s_1}{h}F_1(\hat{y}) + 210\frac{s_2}{h}F_2(\hat{y}) \\
& + \frac{434}{h}\left(s_3 - \sum_{i=4}^{9} s_i\right)F_3(\hat{y}) + \sum_{i=4}^{9} \frac{1}{\displaystyle\int_0^1 F_i(\hat{y})\,\mathrm{d}\hat{y}} \frac{s_i}{h} F_i(\hat{y}).
\end{aligned} \tag{4.24}
$$

As will be shown below, the explicit formulations of the polynomials F_i, $4 \leqslant i \leqslant 9$, will not be required so that in practice the Gram–Schmidt orthogonalization procedure is limited to the determination of F_1, F_2 and F_3.

We now apply a Galerkin projection. Let us consider in detail the first four residuals for the momentum equation. Being of $O(\epsilon^2)$ or higher, the corrective fields s_i, $4 \leqslant i \leqslant 9$, may contribute only through the evaluation of the zeroth-order viscous term $\int_0^h F_i(y/h)\partial_{yy}u\,\mathrm{d}y$, which after integrating twice by parts becomes $\int_0^h F_i''(y/h)u\,\mathrm{d}y$. Notice that $F_0'' = -1$, $F_1'' = 14F_0 - \frac{17}{3}$, $F_2'' = \frac{1485}{28}F_1 + \frac{909}{28}F_0 - 13$ and $F_3'' = \frac{88803}{868}F_1 + \frac{31779}{868}F_0 - \frac{531}{31}$, are linear combinations of 1, F_0 and F_1. Consequently and making use of the orthogonality of the polynomials F_i, the first four residuals of the momentum equation $\mathcal{R}_q(F_i)$, $0 \leqslant i \leqslant 3$, do not involve the second-order fields s_i, $i \geqslant 4$. After some algebraic manipulations, they lead to a set of evolution equations for q, s_1, s_2, s_3 which has the formal expression

$$
\partial_t U_{\mathrm{q}} = M_{\mathrm{q}} V_{\mathrm{q}} \tag{4.25}
$$

where $U_{\mathrm{q}} = (q, s_1, s_2, s_3)^t$, V_{q} is a 20×1 vector whose components depend on h, q, θ, $s_{1,2,3}$ and their space derivatives and M_{q} is a 4×20 matrix with constant coefficients.

The same argument applies to the temperature fields so that the set of test functions G_i must be completed at second order with five polynomials of degree up to nine. Nevertheless, since G_i'', $0 \leqslant i \leqslant 3$, are not linear combinations of G_i, $0 \leqslant i \leqslant 3$, the first four residuals do not form a closed set of equations for θ, t_1, t_2 and t_3. Yet, a basis for the set of polynomials of degree up to five satisfying the Dirichlet condition at the wall can be obtained by introducing only one polynomial orthogonal to the first four G_i. This polynomial G_4 is given explicitly by

$$
G_4(\hat{y}) = \hat{y} - \frac{128}{15}\hat{y}^2 + 24\hat{y}^3 - \frac{192}{7}\hat{y}^4 + 11\hat{y}^5. \tag{4.26}
$$

The temperature field can now be written at second-order as

$$
\begin{aligned}
T = {} & 1 + (\theta - 1 - t_1 - t_2 - t_3 - t_4)G_0(\hat{y}) - \frac{3}{2}t_1 G_1(\hat{y}) + \frac{5}{2}t_2 G_2(\hat{y}) \\
& - \frac{15}{4}\left(t_3 - \sum_{i=5}^{8} t_i\right)G_3(\hat{y}) + \frac{105}{4}t_4 G_4(\hat{y}) + \sum_{i=5}^{8} t_i \frac{G_i(\hat{y})}{G_i(1)}.
\end{aligned} \tag{4.27}
$$

This formulation ensures that the evaluation of $\int_0^h G_i''(\hat{y})T\,dy$, $0 \leqslant i \leqslant 4$ does not require the definitions of G_i, $i \geqslant 5$. By applying next a Galerkin projection to the heat equation, the first five residuals $\mathscr{R}_T(G_i)$, $0 \leqslant i \leqslant 4$, constitute a closed set. Since the amplitude t_4 is of $O(\epsilon^2)$, its space and time derivatives can be neglected at this order, so that an explicit formulation as function of h, θ, t_1, t_2 and t_3 can be obtained, thus expressing the slaving of the former to the latter. After some tedious algebraic manipulations, one obtains a set of evolution equations for θ, t_1, t_2, t_3 which can be written formally as

$$Pr\partial_t \boldsymbol{U}_T = \boldsymbol{M}_T \, \boldsymbol{V}_T \qquad (4.28)$$

where $\boldsymbol{U}_T = (\theta, t_1, t_2, t_3)^t$, $\boldsymbol{V}_T$ is a 21×1 vector whose components depend on h, q, θ, $s_{1,2,3}$ and their space derivatives and $\boldsymbol{M}_T$ is a 4×21 matrix with constant coefficients. Hence we finally have a set of nine coupled evolution equations, namely (4.3), (4.25), (4.28) for nine unknowns, given in Appendix A.

5. Reduced models

However, our full-size second-order model, derived from the systematic procedure outlined in the previous section, is of little use because of its complexity. It is hence necessary to obtain models of reduced dimensionality which also retain the dynamic characteristics of the full-size model.

A significant reduction can be achieved by expanding our unknowns in series of ϵ and performing an appropriate gradient expansion of the full-size model (4.3), (4.25), (4.28), thus writing formally $q = q^{(0)} + q^{(1)} + \cdots$, $\theta = \theta^{(0)} + \theta^{(1)} + \cdots$, $s_i = s_i^{(1)} + s_i^{(2)} + \cdots$, $t_i = t_i^{(1)} + t_i^{(2)} + \cdots$, where the superscript denotes the order of differentiation with respect to x. At $O(\epsilon^0)$, we recover the flat-film solution with $q^{(0)} = h^3/3$ and $\theta^{(0)} = 1/(1 + Bih)$. Inserting $q = q^{(0)}$ into the kinematic equation $\partial_t h + \partial_x q = 0$ yields a single evolution equation for the film thickness (Benney 1966). Because the heat transfer and the mechanical equilibrium of the flat film are two decoupled problems in this limit, this equation does not involve the Marangoni effect that appears at first order through the terms $Ma\partial_x\theta$ in the system (4.25) – or the continuity of the tangential stress (3.6). At $O(\epsilon)$, we obtain the first-order correction to the flow rate,

$$q^{(1)} = \left(\frac{2}{15}h^6 - \frac{\cot\beta}{3}h^3 + BiMa\frac{h^2}{2(1 + Bih)^2} \right) \partial_x h, \qquad (5.1)$$

through the expansion of (4.25) and utilizing the expression for $\theta^{(0)}$. Then, the correction to the leading-order temperature field, $\theta^{(1)} = Bi\,Prh^4\partial_x h(7Bih - 15)/[120(1 + Bih)^3]$, is similarly provided by the expansion of (4.28). Finally, the second-order evolution equation for the free surface is

$$\partial_t h + \partial_x \left(q^{(0)} + q^{(1)} + q^{(2)} + \frac{\Gamma}{3}h^3\partial_{xxx}h \right) = 0, \qquad (5.2)$$

where the main contribution of the surface tension – through the corresponding term of $O(\epsilon^3)$ – has been kept since, as we have already pointed out, it is the only term that prevents the waves from breaking. The expression for $q^{(2)}$ is obtained from (4.25) without having to solve for the gradient expansion up to second order of the

temperature-related fields θ and t_i:

$$q^{(2)} = \left(\frac{127}{315}h^9 - \frac{8}{15}\cot\beta\, h^6 + \frac{7}{3}h^3 \right.$$

$$+ MaBih^5 \frac{6(5Pr + 11) - 5Bi(2Pr - 15)h + Bi^2(-7Pr + 9)h^2}{120(1 + Bih)^4} \bigg) (\partial_x h)^2$$

$$+ \left(\frac{4}{63}h^{10} - \frac{10}{63}\cot\beta\, h^7 + h^4 + MaBih^6 \frac{15Pr + 57 + Bi(-7Pr + 57)h}{240(1 + Bih)^3} \right) \partial_{xx}h.$$

$$(5.3)$$

Since the derivation of the system of equations (4.25), (4.28) is coherent up to $O(\epsilon^2)$, so that no terms of $O(\epsilon^2)$ or smaller have been omitted, the above gradient expansion does agree exactly with the Benney expansion at $O(\epsilon^2)$. The JDB equation derived by Joo *et al.* (1991) is then recovered by simply neglecting the second-order terms $q^{(2)}$ in (5.2).

However, as we have already pointed out in the Introduction, it is well known that Benney-type evolution equations exhibit non-physical finite-time blow-up behaviour when these equations are integrated in time and for sufficiently large Reynolds numbers. Therefore, the aim here is to obtain a prototype set of equations of reduced dimensionality without the drawbacks of Benney's single evolution equation for the film thickness in (5.2) and, hence, of higher degree of complexity than Benney's expansion. At the same time, it should be of lower degree of complexity than the full-size system with nine unknowns in (4.3), (4.25), (4.28). This reduced model should fully resolve conditions near criticality and hence not only correct all critical quantities but also give the full long-wave lubrication equation with an appropriate expansion. Finally, the reduced model should also capture the second-order dissipative effects and hence it should accurately describe the dynamics of the film up to moderate Reynolds and Péclet numbers.

For this purpose, we consider the projections for the velocity and temperature fields given by (4.24) and (4.27). The corrective fields s_i and t_i correspond to polynomials of increasing degrees and, hence, they exhibit increasingly abrupt variations. Therefore, viscosity and thermal diffusivity will tend to damp them. This can be shown for example by linearizing the model equations around the flat-film solution assuming no spatial dependence of the perturbations, i.e. set the wavenumber equal to zero. With this hypothesis, $\mathrm{d}h/\mathrm{d}t = 0$ and the film thickness is constant. Furthermore, both systems (4.25) and (4.28) are decoupled and writing $q = h_N^3(1/3 + \varepsilon\tilde{q})$, $s_i = \varepsilon h_N^3 \tilde{s}_i$, $\theta = (1 + \varepsilon\tilde{\theta})/(1 + Bih_N)$, $t_i = \varepsilon\tilde{t}_i$ with $\varepsilon \ll 1$, one obtains two linear systems in the form

$$Re\frac{\mathrm{d}V}{\mathrm{d}t} = A\,V, \quad PrRe\frac{\mathrm{d}W}{\mathrm{d}t} = B\,W, \quad (5.4)$$

where $V = (\tilde{q}, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3)^t$, $W = (\tilde{\theta}, \tilde{t}_1, \tilde{t}_2, \tilde{t}_3)^t$ and A and B are two square matrices of dimensions 4×4. The eigenvalues of A and B are -63.6, -26.6, -7.42, -0.82 and -89.1, -21.0, -7.40, -0.82, respectively. Therefore, there is a large gap between the least stable (largest) eigenvalues and the other eigenvalues. The spectra are hence well separated and the perturbations associated with the eigenvalues far from zero are quickly damped. The dynamics of the flow in the limit of long waves is therefore dominated by the eigenvectors corresponding to the eigenvalues closest to zero. These are $(\tilde{q}, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3)^t = (-1.00, 1.33 \times 10^{-2}, -1.38 \times 10^{-4}, 2.22 \times 10^{-7})^t$ and $(\tilde{\theta}, \tilde{t}_1, \tilde{t}_2, \tilde{t}_3)^t = (0.976, -0.219, 8.08 \times 10^{-3}, 7.52 \times 10^{-4})^t$. In both eigenvectors, the coefficients corresponding to the corrections $\tilde{s}_i$ and $\tilde{t}_i$ are negligible except for $\tilde{t}_1$,

which is however four times smaller than the coefficient corresponding to $\tilde{\theta}$. It can then be deduced that even if nine amplitudes h, q, θ, s_i and t_i, $1 \leqslant i \leqslant 3$, are needed to describe the dynamics of the flow at second order, only q, h and θ will play a significant role and the other ones will virtually be slaved to their dynamics, at least for some range of Reynolds numbers. Therefore, it seems possible to develop a reduced model in terms of h, q and θ that only reproduces reliably the dynamics of the film up to moderate Reynolds and Péclet numbers.

Let us now consider the two residuals corresponding to the parabolic velocity profile, $\mathcal{R}_q(F_0)$, and to the linear temperature distribution, $\mathcal{R}_T(G_0)$. These two residuals do not involve the fields s_i and t_i directly but only their space and time derivatives, except in products with derivatives of h or q. Therefore, the fields s_i and t_i can be eliminated at second order provided that explicit expressions for them as functions of h, q and θ and their derivatives, are available at first order. Such relations can easily be obtained by truncating at first order the eight evolution equations (4.25) and (4.28). Solving a system of eight equations for six unknowns then leads to two compatibility conditions which correspond to the first-order model (4.18). Thus, the expressions for the fields s_i and t_i as functions of h, q and θ are not unique and can be modified with the help of the two compatibility conditions. Here, we will choose to express the fields s_i and t_i through the derivatives of h, q and θ only, so that we can obtain reduced models whose first-order terms correspond to the first-order model (4.18). The final expressions for the fields s_i and t_i are given in Appendix B. Substituting these expressions into the residuals of the momentum and heat equations, $\mathcal{R}_q(F_0)$ and $\mathcal{R}_T(G_0)$, corresponding to a parabolic and a linear weight, respectively, and making use of the kinematic relation $\partial_t h = -\partial_x q$ yields

$$
\begin{aligned}
\partial_t q = {}& \frac{5}{6}h - \frac{5}{2}\frac{q}{h^2} - \frac{17}{7}\frac{q}{h}\partial_x q + \left(\frac{9}{7}\frac{q^2}{h^2} - \frac{5}{6}\cot\beta\, h\right)\partial_x h \\
& + 4\frac{q}{h^2}(\partial_x h)^2 - \frac{9}{2h}\partial_x q\,\partial_x h - 6\frac{q}{h}\partial_{x^2}h + \frac{9}{2}\partial_{x^2}q \\
& - \frac{5}{4}Ma\partial_x\theta + \frac{5}{6}\Gamma h\partial_{xxx}h + \mathrm{Ine}_q[h,\,q,\,\theta] + Ma\mathrm{Mar}_q[h,\,q,\,\theta], \quad (5.5a)
\end{aligned}
$$

$$
\begin{aligned}
Pr\partial_t\theta = {}& 3\frac{(1 - \theta - Bih\theta)}{h^2} + Pr\left[\frac{7}{40}\frac{(1-\theta)}{h}\partial_x q - \frac{27}{20}\frac{q}{h}\partial_x\theta\right] \\
& + \left(1 - \theta - \frac{3}{2}Bih\theta\right)\left(\frac{\partial_x h}{h}\right)^2 + \frac{\partial_x h\partial_x\theta}{h} + (1-\theta)\frac{\partial_{x^2}h}{h} + \partial_{x^2}\theta \\
& + Pr\,\mathrm{Ine}_{q\theta}[h,\,q,\,\theta] + Pr^2\,\mathrm{Ine}_{\theta}[h,\,q,\,\theta] + MaPr\,\mathrm{Mar}_{\theta}[h,\,q,\,\theta], \quad (5.5b)
\end{aligned}
$$

where hidden in Ine_q, Mar_q, $\mathrm{Ine}_{q\theta}$, Ine_{θ} and Mar_{θ}, are the second-order inertial terms induced by the corrections to the flat-film solution (B 1). Ine_q contains terms of the momentum equation produced by the advection of the first-order corrections of the velocity profile (B 1a)–(B 1c). Mar_q denotes the terms of the momentum equation associated with the Marangoni flow produced by the gradient of temperature at the free surface. Similarly, Ine_{θ} contains inertial terms originating from the averaged heat equation through the advection of the first-order corrections of the temperature profile (B 1d)–(B 1). The terms contained in $\mathrm{Ine}_{q\theta}$ and Mar_{θ} originate from the advection of the linear flat-film temperature distribution by the first-order corrections of the velocity profile induced by the deformation of the free surface and the Marangoni flow, respectively.

6. Regularized reduced model

Although the explicit formulations of Ine_q, $\mathrm{Ine}_{q\theta}$, Ine_θ, Mar_q and Mar_θ are complicated and involve time derivatives, they can be drastically simplified using the relations provided by the zeroth-order flat-film solution (2.9)

$$q = \frac{h^3}{3} + O(\epsilon) \quad \text{and} \quad \theta = \frac{1}{1 + Bi\, h} + O(\epsilon). \tag{6.1}$$

Therefore, the second-order terms appearing in (5.5) do not have a unique formulation since a large number of asymptotically equivalent expressions is possible by using (6.1) (note, however, that the full-size second-order model given in Appendix A is unique). Moreover, as already pointed out, the first-order expressions for the fields s_i, t_i as provided in (B 1) are not unique and can be modified with the help of the two compatibility conditions given by the first-order model (4.18). Hence, we do not end up with a single model fully compatible with the Benney long-wave expansion up to second order (5.1)–(5.3) but with a whole family of such models. Nevertheless, even if all of them are asymptotically equivalent, they might not necessarily behave in the same way if the assumptions leading to their formulation are violated. Indeed, the methodology presented here is based primarily on the assumption that inertia plays effectively a 'secondary' role, with all inertial terms being at least first order in the film parameter. Yet, in practical applications, both Reynolds and Péclet numbers are in general large and a formulation that elucidates the way both H- and S-modes interact in the high-Reynolds/Péclet-number flow regimes remains an open question.

In the context of isothermal film flows, the derivation of the Benney equation also requires the assumption of a perturbative role for inertia. As a consequence and as we have pointed out several times already, at sufficiently large Reynolds numbers, a non-physical catastrophic behaviour of the non-stationary solutions leading to blow-ups in finite time is observed. The occurrence of finite-time blow-ups has been shown to be closely related to the disappearance of the single-hump solitary wave solutions (Pumir *et al.* 1983). Ooshida's regularization procedure of the Benney expansion on the other hand leads to a single evolution equation for the free surface h that does not exhibit this severe drawback (Ooshida 1999). Nevertheless, the Ooshida equation fails to describe accurately the dynamics of the film at moderate Reynolds numbers as its solitary wave solutions exhibit unrealistically small amplitudes and speeds.

Another single evolution equation including the second-order dissipation effects was recently introduced by Panga & Balakotaiah (2003). The inertial terms appearing in the model equations offered by both Ooshida (1999) and Panga & Balakotaiah (2003) can be shown to be equivalent to each other by using the lowest-order expression $\partial_t h = -h^2 \partial_x h$ provided by the flat-film solution (6.1) and the mass conservation equation. Thus, Panga & Balakotaiah's formulation can be modified such that its inertial terms correspond to Ooshida's equation. This simple procedure was shown to cure the non-physical loss of the solitary wave solutions and thus to avoid the occurrence of finite-time blow-ups (Ruyer-Quil & Manneville 2004).

In addition, Ooshida identified two regimes in the solitary wave solution branch. The first one, the so-called *drag–gravity* regime, corresponds to the balance of the gravitational acceleration with the viscous drag and with inertia playing only a perturbative role. This regime is observable near the instability threshold or for low-amplitude waves. The second one appears at larger Reynolds numbers and corresponds to a noticeable jump in the speed and amplitude of the observed solitary waves. In this *drag–inertia* regime, inertia plays a dominant role and therefore the basic assumption of a perturbative role of inertia is clearly violated there. Yet, the

construction of the solitary wave solutions of Ooshida's equation proved that, even if a quantitative agreement cannot be achieved in the drag-inertia regime using a single evolution equation, a qualitative description of the wave dynamics is still possible. Moreover, comparisons of the wave characteristics obtained using the boundary-layer approximation or the Shkadov model with two evolution equations for h and q are in good agreement even for large-amplitude waves (Chang, Demekhin & Kopelevitch 1993). A similar study using a simplified second-order Galerkin model corresponding to (4.3), (5.5a) with $Ma = 0$ and $\mathrm{Ine}_q = 0$ was shown to correctly reproduce experimental wave shapes and speeds even up to moderate Reynolds numbers, clearly in the drag–inertia regime (Ruyer-Quil & Manneville 2000).

Therefore, it seems possible to select at least one formulation of the reduced second-order model having the form (4.3), (5.5) that will enable us to describe at least qualitatively – and possibly quantitatively – the wave dynamics in the drag–inertia regime. Two simple tests for the validity of the different models are: (i) the linear stability of the basic flow solution and its comparison to the Orr–Sommerfeld analysis of the linearized Navier–Stokes/energy equations, (ii) the construction of the single-hump solitary wave family to ensure the existence of solitary waves for the largest possible range of parameters thus preventing the occurrence of non-physical blow-ups.

Consider now the averaged momentum equation (5.5a). The second-order inertial terms Ine_q and Mar_q must remain small compared to the first-order ones to ensure convergence of the perturbation scheme. However, this might not be the case if the Reynolds/Marangoni numbers are large enough since the order of the nonlinearities contained in Ine_q and Mar_q is high. A simple way to achieve the smallness of Ine_q and Mar_q compared to the first-order terms for the largest possible range of parameters is to express the second-order terms in a form as close as possible to the first-order ones. The first-order inertial terms of (5.5a) are $\partial_t q$ on the left-hand side and $-\frac{17}{7}(q/h)\partial_x q$ and $\frac{9}{7}(q^2/h^2)\partial_x h$ on the right-hand side. Their coefficients are uniform in space which suggests that inertia acts similarly everywhere on the plane. It might not be so, however, since the velocity profile is modified by the thermocapillary effect and the deformation of the interface. This then suggests postulating coefficients that depend slowly on time and position, or more precisely on the local gradient of the film thickness and local properties. Based on this consideration, we may define a local reduced Reynolds number as $q\partial_x h$, which compares the local inertial time $t_0 q|\partial_x h|/h^2$ to the local relaxation time $\nu/(l_0^2 h^2)$. Notice that $q\partial_x h$ has the sign of the local slope $\partial_x h$ and therefore inertial corrections to the velocity profiles are assumed to act differently at the front and the back of a propagating solitary hump. Let us then consider a solitary hump in its frame of reference. With the wave moving faster than the fluid, in the moving frame of reference the liquid flows from the right to the left such that it goes upwards at the front of the wave and downwards at its back. Second-order inertia effects tend to increase the transport of momentum to the crest at the front of the wave, which is in favour of the instability, and decrease it at its back. In the same spirit, let us define a local Marangoni number $Ma\partial_x\theta/h$ comparing the local thermocapillary stress $-\mathrm{d}\sigma/\mathrm{d}T|_{T_0}(T_\mathrm{w} - T_\mathrm{a})\partial_x\theta/l_0$ to the hydrostatic pressure $\rho g l_0 h$. The second-order inertial terms must therefore have the form

$$\mathrm{Ine}_q \propto q\,\partial_x h \left[\partial_t q - \frac{9}{7}\frac{q^2}{h^2}\partial_x h + \frac{17}{7}\frac{q}{h}\partial_x q \right], \qquad (6.2a)$$

$$\mathrm{Mar}_q \propto \frac{1}{h}\partial_x \theta \left[\partial_t q - \frac{9}{7}\frac{q^2}{h^2}\partial_x h + \frac{17}{7}\frac{q}{h}\partial_x q \right]. \qquad (6.2b)$$

To obtain the precise expressions for Ine_q and Mar_q, we use an asymptotic argument and compel the postulated expressions to agree with the expressions obtained by substituting the first-order relations for the fields s_i and t_i in the residuals $\mathscr{R}_q(F_0)$ and $\mathscr{R}_T(G_0)$ (see the analysis in the previous section):

$$\mathrm{Ine}_q = \frac{1}{210}h^2\partial_{tt}q + \frac{17}{630}hq\,\partial_{xt}q - \frac{1}{105}q\partial_x h\partial_t q$$
$$+ \frac{1}{42}h\partial_x q\,\partial_t q - \frac{26}{231}\frac{q^2\partial_x h\partial_x q}{h} + \frac{653}{8085}q(\partial_x q)^2$$
$$+ \frac{386}{8085}q^2\partial_{xx}q + \frac{104}{2695}\frac{q^3(\partial_x h)^2}{h^2} - \frac{78}{2695}\frac{q^3\partial_{xx}h}{h}, \tag{6.3a}$$

$$\mathrm{Mar}_q = \frac{5}{112}q\partial_x h\partial_x\theta + \frac{19}{336}h\partial_x q\partial_x\theta + \frac{1}{48}h^2\partial_{xt}\theta + \frac{15}{224}hq\partial_{xx}\theta. \tag{6.3b}$$

Utilizing the zeroth-order equivalence (6.1), the right-hand side side of (6.3a) is asymptotically equivalent to $-\frac{1}{630}h^7(\partial_x h)^2$ and $\partial_t q - \frac{9}{7}\frac{q^2}{h^2}\partial_x h + \frac{17}{7}\frac{q}{h}\partial_x q$ to $-\frac{1}{3}h^4\partial_x h$. Therefore we obtain

$$\mathrm{Ine}_q = \frac{1}{70}q\partial_x h\left(\partial_t q - \frac{9}{7}\frac{q^2}{h^2}\partial_x h + \frac{17}{7}\frac{q}{h}\partial_x q\right). \tag{6.4}$$

Consider now the second-order terms Mar_q induced by the correction to the velocity profile due to the Marangoni effect. It turns out that it is not possible to obtain an expression in the form of (6.2b) which is asymptotically equivalent to the right-hand side of (6.3b). This is due to the presence of the last two terms in (6.3b), $\frac{1}{48}h^2\partial_{xt}\theta$ and $\frac{15}{224}hq\partial_{xx}\theta$. Nevertheless, the expression

$$\mathrm{Mar}_q = \frac{5}{56}\frac{1}{h}\partial_x\theta\left(\partial_t q - \frac{9}{7}\frac{q^2}{h^2}\partial_x h + \frac{17}{7}\frac{q}{h}\partial_x q\right) + \frac{1}{224}qh\partial_{xx}\theta, \tag{6.5}$$

is asymptotically equivalent to (6.3b). Note that although the last term of this expression, $\frac{1}{224}qh\partial_{xx}\theta$, cannot be eliminated – the remaining terms are of the form suggested by (6.2b) – this term contains nonlinearities of smaller order than the nonlinearities that can cause blow-up (Hocherman & Rosenau 1993; Bertozzi & Pugh 1998).

Other formulations have been tested and compared using both the linear and nonlinear tests proposed above. In view of the results and considering the second-order inertial and thermocapillary terms appearing in the averaged heat balance (5.5b), Ine_θ, $\mathrm{Ine}_{q\theta}$ and Mar_θ (induced by the deviations of the velocity and temperature profiles from the flat-film Nusselt solution), it has not been possible to obtain for these terms asymptotically equivalent formulations analogous to (6.4) and (6.5), if the temperature field is assumed to be slaved to the free-surface temperature θ only. This failure suggests describing the temperature field by allowing at least the first correction t_1 to θ to have its own dynamics. Such an approach, however, is beyond the scope of the present study and will be presented in a future paper. Therefore, $\mathrm{Ine}_{q\theta}$, Ine_θ and Mar_θ will all be set equal to zero. This assumption is still consistent with the gradient expansion at second order since the interfacial temperature is only coupled to the local flow rate through its gradient (already of $O(\epsilon)$). Yet, Ine_q and Mar_q do contribute to $q^{(2)}$ and should be kept. The final proposed model

therefore is

$$\partial_t h = -\partial_x q, \tag{6.6a}$$

$$\partial_t q = \frac{9}{7}\frac{q^2}{h^2}\partial_x h - \frac{17}{7}\frac{q}{h}\partial_x q$$

$$+ \left\{ \frac{5}{6}h - \frac{5}{2}\frac{q}{h^2} + 4\frac{q}{h^2}(\partial_x h)^2 - \frac{9}{2h}\partial_x q\,\partial_x h - 6\frac{q}{h}\partial_{xx}h + \frac{9}{2}\partial_{xx}q \right.$$

$$\left. - \frac{5}{6}\cot\beta\, h\partial_x h + \frac{5}{6}\Gamma h\partial_{xxx}h - Ma\left(\frac{5}{4}\partial_x\theta - \frac{1}{224}hq\partial_{xx}\theta\right) \right\}$$

$$\times \left(1 - \frac{1}{70}q\partial_x h + Ma\frac{5}{56h}\partial_x\theta \right)^{-1}, \tag{6.6b}$$

$$Pr\partial_t\theta = 3\frac{(1-\theta-Bih\theta)}{h^2} + Pr\left[\frac{7}{40}\frac{(1-\theta)}{h}\partial_x q - \frac{27}{20}\frac{q}{h}\partial_x\theta \right]$$

$$+ \left(1 - \theta - \frac{3}{2}Bih\theta \right)\left(\frac{\partial_x h}{h} \right)^2 + \frac{\partial_x h\partial_x\theta}{h} + (1-\theta)\frac{\partial_{x^2}h}{h} + \partial_{xx}\theta. \tag{6.6c}$$

We shall refer to this as the *reduced regularized model* since it ensures the smallness of Ine_q and Mar_q compared to the corresponding first-order terms, thus avoiding unrealistic blow-up behaviour. In Part 2 we shall demonstrate that this model satisfies the linear and nonlinear criteria and, as expected, its gradient expansion leads to the free-surface equation (5.2) where the expressions (5.1) and (5.3) for $q^{(1)}$ and $q^{(2)}$ are exactly recovered.

7. Conclusion

We have considered the dynamics of a thin film falling down a uniformly heated plane. A systematic methodology for the derivation of a set of nonlinear partial differential equations that describe the evolution of the film was developed. This methodology is a combination of the classical long-wave theory with polynomial expansions for the velocity and temperature fields followed by a weighted residuals approach.

The large dimensionality of the set of equations that describe the evolution of the film (nine for the full-size second-order model), necessitated the development of a procedure for the reduction of its dimension. This led to a family of (three-equation) models compatible with the classical gradient expansion up to second order. One of these models, referred to as the 'regularized reduced model', was obtained by postulating that the formulation of the second-order inertial terms should remain as close as possible to the first-order ones. This procedure effectively regularizes the second-order terms as it ensures that these terms are well-behaved for the largest possible range of parameters. Different formulations of the second-order inertial terms have also been tried (not shown). The regularized reduced model (6.6) offers the best comparisons with the Orr–Sommerfeld analysis and its one-humped solitary wave solutions exist for all Reynolds numbers, as will be demonstrated in Part 2.

Our procedure, based on a gradient expansion combined with a Galerkin projection with polynomial test functions, was made possible by the simple zeroth-order solution (2.8) which corresponds precisely to polynomial velocity and temperature profiles. These features can also be found in the case of flow in Hele-Shaw cells of thin gaps (Ruyer-Quil 2001). Similar favourable circumstances are quite rare. Yet, because

other geometries can be connected to the 'flat-film case' with the help of appropriate expansions, we believe that the procedure outlined here can be successfully applied to obtain low-dimensional models in other situations. These might include thin films in the presence of surfactants (Shkadov, Velarde & Shkadova 2004) and chemical reactions (Trevelyan & Kalliadasis 2004*a*, *b*).

Appendix A. Full-size second-order model

$$
\begin{aligned}
\partial_t q = {} & \frac{30}{31}h - \frac{90}{31}\frac{q}{h^2} - \frac{1050}{31}\frac{s_1}{h^2} - \frac{3690}{31}\frac{s_2}{h^2} - \frac{9066}{31}\frac{s_3}{h^2} - \frac{12}{5}\frac{q\partial_x q}{h} + \frac{6}{5}\frac{q^2\partial_x h}{h^2} \\
& - \frac{30}{31}\cot\beta\, h\partial_x h - \frac{213}{248}Ma\partial_x\theta + \frac{1569}{248}\frac{q(\partial_x h)^2}{h^2} - \frac{12}{5}\frac{qs_1\partial_x h}{h^2} - \frac{4248}{2015}\frac{qs_2\partial_x h}{h^2} \\
& - \frac{5296}{3875}\frac{qs_3\partial_x h}{h^2} - \frac{1569}{248}\frac{\partial_x h\partial_x q}{h} + \frac{12}{5}\frac{s_1\partial_x q}{h} + \frac{1026}{403}\frac{s_2\partial_x q}{h} + \frac{11722}{3875}\frac{s_3\partial_x q}{h} \\
& + \frac{12}{5}\frac{q\partial_x s_1}{h} + \frac{4626}{2015}\frac{q\partial_x s_2}{h} + \frac{1538}{775}\frac{q\partial_x s_3}{h} + \frac{1069}{248}\partial_{xx}q - \frac{2847}{496}\frac{q\partial_{xx}h}{h} + \frac{30}{31}\Gamma h\partial_{xxx}h,
\end{aligned}
\tag{A 1a}
$$

$$
\begin{aligned}
\partial_t s_1 = {} & \frac{1}{10}h - \frac{3}{10}\frac{q}{h^2} - \frac{126}{5}\frac{s_1}{h^2} - \frac{126}{5}\frac{s_2}{h^2} - \frac{126}{5}\frac{s_3}{h^2} + \frac{1}{35}\frac{q\partial_x q}{h} - \frac{3}{35}\frac{q^2\partial_x h}{h^2} \\
& - \frac{1}{10}\cot\beta\, h\partial_x h + \frac{3}{8}Ma\partial_x\theta + \frac{93}{40}\frac{q(\partial_x h)^2}{h^2} + \frac{108}{55}\frac{qs_1\partial_x h}{h^2} - \frac{5022}{5005}\frac{qs_2\partial_x h}{h^2} \\
& + \frac{6}{35}\frac{qs_3\partial_x h}{h^2} - \frac{69}{40}\frac{\partial_x h\partial_x q}{h} - \frac{103}{55}\frac{s_1\partial_x q}{h} + \frac{9657}{5005}\frac{s_2\partial_x q}{h} - \frac{1}{35}\frac{s_3\partial_x q}{h} - \frac{39}{55}\frac{q\partial_x s_1}{h} \\
& + \frac{10557}{10010}\frac{q\partial_x s_2}{h} + \frac{19}{70}\frac{q\partial_x s_3}{h} - \frac{9}{40}\partial_{xx}q + \frac{21}{80}\frac{q\partial_{xx}h}{h} + \frac{1}{10}\Gamma h\partial_{xxx}h,
\end{aligned}
\tag{A 1b}
$$

$$
\begin{aligned}
\partial_t s_2 = {} & \frac{13}{420}h - \frac{13}{140}\frac{q}{h^2} - \frac{39}{5}\frac{s_1}{h^2} - \frac{11817}{140}\frac{s_2}{h^2} - \frac{11817}{140}\frac{s_3}{h^2} - \frac{13}{420}\cot\beta\, h\partial_x h - \frac{13}{64}Ma\partial_x\theta \\
& - \frac{3211}{4480}\frac{q(\partial_x h)^2}{h^2} - \frac{4}{11}\frac{qs_1\partial_x h}{h^2} + \frac{18}{11}\frac{qs_2\partial_x h}{h^2} - \frac{38}{25}\frac{qs_3\partial_x h}{h^2} + \frac{2613}{4480}\frac{\partial_x h\partial_x q}{h} \\
& - \frac{2}{33}\frac{s_1\partial_x q}{h} - \frac{19}{11}\frac{s_2\partial_x q}{h} + \frac{76}{25}\frac{s_3\partial_x q}{h} + \frac{6}{55}\frac{q\partial_x s_1}{h} - \frac{288}{385}\frac{q\partial_x s_2}{h} + \frac{73}{70}\frac{q\partial_x s_3}{h} \\
& + \frac{559}{2240}\partial_{xx}q - \frac{2847}{8960}\frac{q\partial_{xx}h}{h} + \frac{13}{420}\Gamma h\partial_{xxx}h,
\end{aligned}
\tag{A 1c}
$$

$$
\begin{aligned}
\partial_t s_3 = {} & \frac{3}{868}h - \frac{9}{868}\frac{q}{h^2} - \frac{27}{31}\frac{s_1}{h^2} - \frac{8181}{868}\frac{s_2}{h^2} - \frac{158709}{868}\frac{s_3}{h^2} - \frac{3}{868}\cot\beta\, h\partial_x h \\
& + \frac{435}{1984}Ma\partial_x\theta + \frac{3}{868}\Gamma h\partial_{xxx}h + \frac{19953}{27776}\frac{q(\partial_x h)^2}{h^2} - \frac{342}{2015}\frac{qs_2\partial_x h}{h^2} + \frac{9894}{3875}\frac{qs_3\partial_x h}{h^2}
\end{aligned}
$$

$$- \frac{19023}{27776} \frac{\partial_x h \partial_x q}{h} - \frac{171}{2015} \frac{s_2 \partial_x q}{h} - \frac{9358}{3875} \frac{s_3 \partial_x q}{h} + \frac{171}{2821} \frac{q \partial_x s_2}{h} - \frac{13653}{10850} \frac{q \partial_x s_3}{h}$$

$$- \frac{2973}{13888} \partial_{xx} q + \frac{17517}{55552} \frac{q \partial_{xx} h}{h}, \tag{A 1d}$$

$$Pr \partial_t \theta = \frac{30009}{1273} \left(\frac{1 - (1 + Bih)\theta}{h^2} \right) - \frac{130950}{1273} \frac{t_1}{h^2} - \frac{1384362}{6365} \frac{t_2}{h^2} - \frac{1863792}{6365} \frac{t_3}{h^2}$$

$$+ Pr \left[- \frac{3117}{203680} \frac{(1-\theta)\partial_x q}{h} - \frac{155877}{101840} \frac{q \partial_x \theta}{h} - \frac{3117}{81472} \frac{t_1 \partial_x q}{h} + \frac{801101}{4480960} \frac{t_2 \partial_x q}{h} \right.$$

$$- \frac{1768473}{1323920} \frac{t_3 \partial_x q}{h} + \frac{3117}{40736} \frac{q \partial_x t_1}{h} - \frac{364701}{2240480} \frac{q \partial_x t_2}{h} + \frac{7840671}{14563120} \frac{q \partial_x t_3}{h}$$

$$- \frac{21819}{101840} \frac{(1-\theta)\partial_x s_1}{h} - \frac{10066647}{21182720} \frac{(1-\theta)\partial_x s_2}{h} - \frac{949089}{8147200} \frac{(1-\theta)\partial_x s_3}{h}$$

$$\left. + \frac{245511}{50920} \frac{s_1 \partial_x \theta}{h} - \frac{44102157}{10591360} \frac{s_2 \partial_x \theta}{h} + \frac{295506139}{4073600} \frac{s_3 \partial_x \theta}{h} \right]$$

$$+ \frac{27463}{1273} \frac{(1-\theta)(\partial_x h)^2}{h^2} - \frac{30009}{2546} Bi \frac{\theta(\partial_x h)^2}{h} + \frac{27463}{1273} \frac{\partial_x h \partial_x \theta}{h}$$

$$+ \frac{(1-\theta)\partial_{xx} h}{h} + \partial_{xx} \theta, \tag{A 1e}$$

$$Pr \partial_t t_1 = \frac{8176}{1273} \left(\frac{1 - (1 + Bih)\theta}{h^2} \right) - \frac{40880}{1273} \frac{t_1}{h^2} - \frac{34874}{1273} \frac{t_2}{h^2} - \frac{28784}{1273} \frac{t_3}{h^2}$$

$$+ Pr \left[- \frac{97063}{407360} \frac{(1-\theta)\partial_x q}{h} - \frac{56327}{203680} \frac{q \partial_x \theta}{h} - \frac{166435}{488832} \frac{t_1 \partial_x q}{h} - \frac{20340413}{26885760} \frac{t_2 \partial_x q}{h} \right.$$

$$- \frac{28729997}{174757440} \frac{t_3 \partial_x q}{h} - \frac{225649}{244416} \frac{q \partial_x t_1}{h} + \frac{222059}{4480960} \frac{q \partial_x t_2}{h} + \frac{16530119}{43689360} \frac{q \partial_x t_3}{h}$$

$$+ \frac{1849169}{1222080} \frac{(1-\theta)\partial_x s_1}{h} + \frac{70164667}{42365440} \frac{(1-\theta)\partial_x s_2}{h} - \frac{11651371}{9776640} \frac{(1-\theta)\partial_x s_3}{h}$$

$$\left. + \frac{294637}{76380} \frac{s_1 \partial_x \theta}{h} - \frac{122474147}{21182720} \frac{s_2 \partial_x \theta}{h} + \frac{88699283}{4888320} \frac{s_3 \partial_x \theta}{h} \right]$$

$$+ \frac{8176}{1273} \frac{(1-\theta)(\partial_x h)^2}{h^2} - \frac{4088}{1273} Bi \frac{\theta(\partial_x h)^2}{h} + \frac{8176}{1273} \frac{\partial_x h \partial_x \theta}{h}, \tag{A1f}$$

$$Pr \partial_t t_2 = \frac{44838}{6365} \left(\frac{1 - (1 + Bih)\theta}{h^2} \right) - \frac{44838}{1273} \frac{t_1}{h^2} - \frac{3231144}{31825} \frac{t_2}{h^2} - \frac{2306304}{31825} \frac{t_3}{h^2}$$

$$+ Pr \left[\frac{158337}{2036800} \frac{(1-\theta)\partial_x q}{h} + \frac{158337}{1018400} \frac{q \partial_x \theta}{h} + \frac{2674149}{5703040} \frac{t_1 \partial_x q}{h} + \frac{13678523}{313667200} \frac{t_2 \partial_x q}{h} \right.$$

$$- \frac{446443197}{509709200} \frac{t_3 \partial_x q}{h} - \frac{1222929}{2851520} \frac{q \partial_x t_1}{h} - \frac{204549663}{156833600} \frac{q \partial_x t_2}{h} - \frac{336704427}{1019418400} \frac{q \partial_x t_3}{h}$$

$$+ \frac{97173}{2036800} \frac{(1-\theta)\partial_x s_1}{h} - \frac{490225113}{211827200} \frac{(1-\theta)\partial_x s_2}{h} - \frac{64692369}{81472000} \frac{(1-\theta)\partial_x s_3}{h}$$

$$\left. - \frac{352833}{254600} \frac{s_1 \partial_x \theta}{h} - \frac{302607783}{105913600} \frac{s_2 \partial_x \theta}{h} + \frac{1304283921}{40736000} \frac{s_3 \partial_x \theta}{h} \right]$$

$$+ \frac{44838}{6365} \frac{(1-\theta)(\partial_x h)^2}{h^2} - \frac{22419}{6365} Bi \frac{\theta(\partial_x h)^2}{h} + \frac{44838}{6365} \frac{\partial_x h \partial_x \theta}{h}, \tag{A1g}$$

$$Pr\partial_t t_3 = \frac{45232}{6365}\left(\frac{1-(1+Bih)\theta}{h^2}\right) - \frac{45232}{1273}\frac{t_1}{h^2} - \frac{2818816}{31825}\frac{t_2}{h^2} - \frac{6293056}{31825}\frac{t_3}{h^2}$$

$$+ Pr\left[-\frac{7579}{254600}\frac{(1-\theta)\partial_x q}{h} - \frac{7579}{127300}\frac{q\partial_x\theta}{h} - \frac{260999}{2138640}\frac{t_1\partial_x q}{h} + \frac{6804757}{10693200}\frac{t_2\partial_x q}{h}\right.$$

$$-\frac{62613409}{139011600}\frac{t_3\partial_x q}{h} + \frac{210079}{1069320}\frac{q\partial_x t_1}{h} - \frac{555439}{1782200}\frac{q\partial_x t_2}{h} - \frac{58821361}{69505800}\frac{q\partial_x t_3}{h}$$

$$-\frac{133699}{381900}\frac{(1-\theta)\partial_x s_1}{h} + \frac{8441321}{26478400}\frac{(1-\theta)\partial_x s_2}{h} + \frac{121848419}{30552000}\frac{(1-\theta)\partial_x s_3}{h}$$

$$\left.-\frac{95509}{190950}\frac{s_1\partial_x\theta}{h} + \frac{55645211}{13239200}\frac{s_2\partial_x\theta}{h} + \frac{284237129}{15276000}\frac{s_3\partial_x\theta}{h}\right]$$

$$+ \frac{45232}{6365}\frac{(1-\theta)(\partial_x h)^2}{h^2} - \frac{22616}{6365}Bi\frac{\theta(\partial_x h)^2}{h} + \frac{45232}{6365}\frac{\partial_x h\partial_x\theta}{h}. \tag{A1h}$$

Appendix B. First-order corrections to the parabolic velocity profile and linear temperature distribution

$$s_1 = \frac{1}{210}h^2\partial_t q - \frac{19}{1925}q^2\partial_x h + \frac{74}{5775}hq\partial_x q + \frac{1}{40}Ma\,h^2\partial_x\theta, \tag{B 1a}$$

$$s_2 = \frac{2}{5775}q^2\partial_x h - \frac{2}{17325}hq\partial_x q - \frac{299}{53760}Ma\,h^2\partial_x\theta, \tag{B 1b}$$

$$s_3 = \frac{5}{3584}Ma\,h^2\partial_x\theta, \tag{B 1c}$$

$$t_1 = Pr\left(\frac{1}{15}h^2\partial_t\theta + \frac{133}{5760}h(\theta-1)\partial_x q + \frac{73}{960}hq\partial_x\theta\right), \tag{B 1d}$$

$$t_2 = Pr\left(-\frac{111}{22400}h(\theta-1)\partial_x q + \frac{79}{11200}hq\partial_x\theta\right), \tag{B 1e}$$

$$t_3 = Pr\left(\frac{1}{3150}h(\theta-1)\partial_x q - \frac{1}{1050}hq\partial_x\theta\right). \tag{B 1f}$$

REFERENCES

ALEKSEENKO, S., NAKORYAKOV, V. & POKUSAEV, B. 1994 *Wave Flow of Liquid Films*, 3rd Edn. Begell House.

BENJAMIN, B. 1957 Wave formation in laminar flow down an inclined plane. *J. Fluid. Mech.* **2**, 554–574.

BENNEY, J. 1966 Long waves on liquid films. *J. Math. Phys.* **45**, 150–155.

BERTOZZI, A. & PUGH, M. 1998 Long-wave instabilities and saturation in thin film equations. *Commun. Pure Appl. Maths* **51**, 625–661.

CHANG, H.-C., DEMEKHIN, E. A. & KOPELEVITCH, D. 1993 Nonlinear evolution of waves on a vertically falling film. *J. Fluid Mech.* **250**, 433–480.

COLINET, P., LEGROS, J. C. & VELARDE, M. G. 2001 *Nonlinear Dynamics of Surface-Tension-Driven Instabilities*, 1st Edn. Wiley-VCH.

DEMEKHIN, E. A., KAPLAN, M. & SHKADOV, V. YA. 1987 Mathematical models of the theory of viscous liquid films. *Izv. Akad. Nauk SSSR, Mekh. Zhidk Gaza* **6**, 73–81.

DEMEKHIN, E. A. & SHKADOV, V. YA. 1984 Three-dimensional waves in a liquid flowing down a wall. *Izv. Akad. Nauk SSSR, Mekh. Zhidk Gaza* **5**, 21–27.

FINLAYSON, B. 1972 *The Method of Weighted Residuals and Variational Principles, with Application in Fluid Mechanics, Heat and Mass Transfer*. Academic.

GOUSSIS, D. & KELLY, R. 1991 Surface wave and thermocapillary instabilities in a liquid film flow. *J. Fluid Mech.* **223**, 25–45, and corrigendum *J. Fluid Mech.* **226**, 663.

HOCHERMAN, T. & ROSENAU, P. 1993 On KS-type equations describing the evolution and rupture of a liquid interface. *Physica* D **67**, 113–125.

JOO, S. W., DAVIS, S. H. & BANKOFF, S. G. 1991 Long-wave instabilities of heated falling films: two-dimensional theory of uniform layers. *J. Fluid Mech.* **230**, 117–146.

KALLIADASIS, S., DEMEKHIN, E. A., RUYER-QUIL, C. & VELARDE, M. 2003a Thermocapillary instability and wave formation on a film flowing down a uniformly heated plane. *J. Fluid Mech.* **492**, 303–338.

KALLIADASIS, S., KIYASHKO, A. & DEMEKHIN, E. A. 2003b Marangoni instability of a thin liquid film heated from below by a local heat source. *J. Fluid Mech.* **475**, 377–408.

KAPITZA, P. L. & KAPITZA, S. P. 1949 Wave flow of thin layers of a viscous fluid. *Zh. Eksper. Teor. Fiz.* **19**, 105; English transl. in *Collected Papers of P. L. Kapitza* (ed. D. T. Haar), pp. 690–709. Pergamon.

OOSHIDA, T. 1999 Surface equation of falling film flows which is valid even far beyond the criticality. *Phys. Fluids* **11**, 3247–3269.

ORON, A., DAVIS, S. H. & BANKOFF, S. G. 1997 Long-scale evolution of thin liquid films. *Rev. Mod. Phys.* **69**, 931–980.

PANGA, M. K. R. & BALAKOTAIAH, V. 2003 Low-dimensional models for vertically falling viscous films. *Phys. Rev. Lett.* **90** (15), 154501.

PEARSON, J. 1958 On convection cells induced by surface tension. *J. Fluid Mech.* **4**, 489–500.

PUMIR, A., MANNEVILLE, P. & POMEAU, Y. 1983 On solitary waves running down an inclined plane. *J. Fluid Mech.* **135**, 27–50.

RUYER-QUIL, C. 2001 Inertial corrections to the Darcy law in a Hele-Shaw cell. *C. R. Acad. Sci. Paris* IIb **329**, 337–342.

RUYER-QUIL, C. & MANNEVILLE, P. 2000 Improved modeling of flows down inclined planes. *Eur. Phys. J.* B **15**, 357–369.

RUYER-QUIL, C. & MANNEVILLE, P. 2002 Further accuracy and convergence results on the modeling of flows down inclined planes by weighted-residual approximations. *Phys. Fluids* **14**, 170–183.

RUYER-QUIL, C. & MANNEVILLE, P. 2004 Comment on "Low-dimensional models for vertically falling viscous films". *Phys. Rev. Lett.* **93** (19), 199401.

SCHEID, B., RUYER-QUIL, C., KALLIADASIS, S., VELARDE, M. G. & ZEYTOUNIAN, R. Kh. 2005 Thermocapillary long waves in a liquid film flow. Part 2. Linear stability and nonlinear waves. *J. Fluid Mech.* **538**, 223–244.

SCHEID, B., RUYER-QUIL, C., THIELE, U., KABOV, O. A., LEGROS, J. C. & COLINET, P. 2005 Validity domain of the Benney equation including Marangoni effect for closed and open flows. *J. Fluid Mech.* **527**, 303–335.

SCRIVEN, L. & STERNLING, C. 1964 On cellular convection driven by surface-tension gradients: effects of mean surface tension and surface viscosity. *J. Fluid Mech.* **19**, 321–340.

SHKADOV, V. YA. 1967 Wave flow regimes of a thin layer of viscous fluid subject to gravity. *Izv. Ak. Nauk SSSR, Mekh. Zhidk. Gaza* **2**, 43–51; English transl. in *Fluid Dyn.* **2**, 29–34.

SHKADOV, V. YA., VELARDE, M. G. & SHKADOVA, V. P. 2004 Falling films and the Marangoni effect. *Phys. Rev.* E **69**, 056310.

SMITH, K. A. 1966 On convective instability induced by surface gradients. *J. Fluid Mech.* **24**, 401–414.

TREVELYAN, P. M. J. & KALLIADASIS, S. 2004a Dynamics of a reactive falling film at large Péclet numbers. I. Long-wave approximation. *Phys. Fluids* **16**, 3191–3208.

TREVELYAN, P. M. J. & KALLIADASIS, S. 2004b Dynamics of a reactive falling film at large Péclet numbers. II. Nonlinear waves far from criticality: Integral-boundary-layer approximation. *Phys. Fluids* **16**, 3209–3226.

VELARDE, M. G. & ZEYTOUNIAN, R. Kh. 2002 *Interfacial Phenomena and the Marangoni Effect*, 1st Edn. Springer.

ZEYTOUNIAN, R. Kh. 1998 The Bénard–Marangoni thermocapillary-instability problem. *Phys.– Uspekhi* **41**, 241–267.